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Analytical solutions for two-level systems with damping

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Abstract
A method is proposed to transform any analytic solution of the Bloch equation into an analytic solution of the Landau–Lifshitz–Gilbert equation. This allows for the analytical description of the dynamics of a two-level system with damping. This method shows that damping turns the linear Schrödinger equation of a two-level system into a nonlinear Schrödinger equation. As an application, the effect of damping on self-induced transparency is investigated.

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1. Introduction

Two-level systems have become almost ubiquitous in modern physics. They are found for instance in laser physics, magnetic resonance spectroscopies, quantum computers, quantum teleportation and optoelectronics. The state of a two-level system can be described by an effective moment $\mathbf{M}$ and its dynamics by the equation $\dot{\mathbf{M}} = -\gamma \mathbf{M} \times \mathbf{B}$, where γ is the gyromagnetic factor and $\mathbf{B}$ is an external time-dependent field. We follow the standard (and inappropriate) custom of calling $\dot{\mathbf{M}} = -\gamma \mathbf{M} \times \mathbf{B}$ the 'Bloch equation'.

The main drawback of the Bloch equation is its absence of damping. There are many phenomenological models of damping for two-level systems. When decoherence is small, the system remains in a pure state, the length of $\mathbf{M}$ is constant and damping is taken into account by the so-called Landau–Lifshitz–Gilbert (LLG) equation [1–3]

$$\dot{\mathbf{M}} = -\gamma \mathbf{M} \times \mathbf{B} + \frac{\alpha}{M} \mathbf{M} \times \dot{\mathbf{M}}, \tag{1}$$

where $M = |\mathbf{M}|$, $\dot{\mathbf{M}} = \mathrm{d}\mathbf{M}/\mathrm{d}t$ and $\alpha > 0$. It was shown that the LLG equation provides a realistic model of ferromagnetic resonance, micromagnetics, spin-valve dynamics [4], the magnetism of thin films [5] and nanomagnets [6] and the dynamics of domain walls in various geometries [7]. Moreover, the LLG equation and the damping parameter can be derived from microscopic models [8, 9].

The linear nature of the Bloch equation allowed for the discovery of many analytic solutions. For example, a recent work identified 26 families of solutions that can be expressed in terms of special functions [10] and significant steps towards the general solution were carried out [11, 12]. Even when analytic solutions are not available, powerful analytic approximation methods exist [13–15].

By contrast, very few solutions of the physically more accurate LLG equation are known. In this paper, we describe a method by which *any* analytic solution of the Bloch equation can be transformed into an analytic solution of the LLG equation. Then, we prove that this transformation turns the linear Schrödinger equation for a two-level system into a nonlinear Schrödinger equation. As an application, we investigate the influence of damping on self-induced transparency.

2. From Bloch to LLG

We describe now the transformation from the solution of the Bloch equation to the solution of the LLG equation. Consider a solution of the Bloch equation $\dot{\mathbf{M}}(\gamma) = -\gamma \mathbf{M}(\gamma) \times \mathbf{B}$, where the dependence of M on the gyromagnetic factor γ is explicit and where $\mathbf{B}$ is a real function of t. Assume now that $\mathbf{M}(\gamma)$ is an *analytic function* of γ. This allows us to define $\mathbf{N} = \mathbf{M}(\bar{\gamma})$, where $\bar{\gamma} = \gamma/(1 - \mathrm{i}\alpha)$, and gives us

$$\dot{\mathbf{N}} = \dot{\mathbf{M}}(\bar{\gamma}) = -\bar{\gamma}\mathbf{M}(\bar{\gamma}) \times \mathbf{B} = -\frac{\gamma}{1 - \mathrm{i}\alpha}\mathbf{N} \times \mathbf{B}.$$

Note that the components of $\mathbf{N}$ are generally complex numbers. The equation of motion implies that the length M of $\mathbf{N}$, defined by $M^2 = \sum_i M_i^2(\bar{\gamma}) = \sum_i N_i^2$, does not depend on t. We consider now the complex number

$$\xi = \frac{N_x + \mathrm{i}N_y}{M + N_z}. \tag{2}$$

If we calculate the derivative of ξ with respect to t, taking account of the fact that M does not depend on time, we find

$$\dot{\xi} = \frac{\dot{N}_x + \mathrm{i}\dot{N}_y}{M + N_z} - \frac{(N_x + \mathrm{i}N_y)\dot{N}_z}{(M + N_z)^2}. \tag{3}$$

If we substitute the equation of motion for $\dot{\mathbf{N}}$, we can check that ξ satisfies

$$\dot{\xi} = \frac{\mathrm{i}\gamma}{2(1 - \mathrm{i}\alpha)}(B^-\xi^2 + 2B_z\xi - B^+), \tag{4}$$

where $B^\pm = B_x \pm \mathrm{i}B_y$. The complex function ξ is used to define the vector $\mathbf{M}'$ by

$$M'_x = \frac{\xi + \xi^*}{|\xi|^2 + 1}M,$$
$$M'_y = -\mathrm{i}\frac{\xi - \xi^*}{|\xi|^2 + 1}M,$$
$$M'_z = \frac{1 - |\xi|^2}{|\xi|^2 + 1}M,$$

so that we still have

$$\xi = \frac{M'_x + \mathrm{i}M'_y}{M + M'_z}. \tag{5}$$

This equation is identical to equation (2), but the components of $\mathbf{M}'/M$ are real, whereas the components of $\mathbf{N}/M$ are usually complex.

We want to determine the equation of motion satisfied by $\mathbf{M}'$. The derivative of $\mathbf{M}'$ with respect to t gives us

$$\dot{M}'_x = \frac{\dot{\xi} - \dot{\xi}(\xi^*)^2 + \dot{\xi}^* - \dot{\xi}^*\xi^2}{(|\xi|^2 + 1)^2} M,$$
$$\dot{M}'_y = -\mathrm{i}\frac{\dot{\xi} + \dot{\xi}(\xi^*)^2 - \dot{\xi}^* - \dot{\xi}^*\xi^2}{(|\xi|^2 + 1)^2} M,$$
$$\dot{M}'_z = -2\frac{\dot{\xi}\xi^* + \xi\dot{\xi}^*}{(|\xi|^2 + 1)^2} M.$$

If we express $\dot{\xi}$ and $\dot{\xi}^*$ through equation (4) and its conjugate, we obtain $\dot{\mathbf{M}}'$ in terms of ξ and ξ^*. If we replace them by equation (5) and its conjugate, we obtain, after a lengthy but straightforward calculation,

$$\dot{M}'_x = -\frac{\gamma}{1+\alpha^2}(B_z M'_y - B_y M'_z) - \frac{\alpha\gamma}{(1+\alpha^2)M} \times \left(B_y M'_x M'_y - B_x M'^2_y + B_z M'_x M'_z - B_x M'^2_z\right),$$
$$\dot{M}'_y = -\frac{\gamma}{1+\alpha^2}(B_x M'_z - B_z M'_x) - \frac{\alpha\gamma}{(1+\alpha^2)M} \times \left(B_z M'_y M'_z - B_y M'^2_z + B_x M'_y M'_x - B_y M'^2_x\right),$$
$$\dot{M}'_z = -\frac{\gamma}{1+\alpha^2}(B_y M'_x - B_x M'_y) - \frac{\alpha\gamma}{(1+\alpha^2)M} \times \left(B_x M'_z M'_x - B_z M'^2_x + B_y M'_z M'_y - B_z M'^2_y\right).$$

This can be rewritten as

$$\dot{\mathbf{M}}' = -\frac{\gamma}{1+\alpha^2}\mathbf{M}' \times \mathbf{B} - \frac{\gamma\alpha}{(1+\alpha^2)M}\mathbf{M}' \times (\mathbf{M}' \times \mathbf{B}).$$

We recognize the Landau–Lifshitz equation in a form equivalent to the LLG equation. To show the equivalence, use the LLG equation to derive $\mathbf{M}' \times \dot{\mathbf{M}}' = -\gamma\mathbf{M}' \times (\mathbf{M}' \times \mathbf{B}) - \alpha M\dot{\mathbf{M}}'$, introduce this expression in the LLG equation and solve for $\dot{\mathbf{M}}'$. Thus, we have transformed a solution $\mathbf{M}$ of the Bloch equation into a solution $\mathbf{M}'$ of the LLG equation (1).

3. Constant external field

We discuss now the simplest example of this transformation, to show how the method works in practice. We consider a constant external magnetic field along the z-axis, i.e. $\mathbf{B}(t) = (0, 0, B_0)$. The solution of the Bloch equation is

$$M_x = M\sin\theta_0\cos(\Omega t + \phi_0),$$
$$M_y = M\sin\theta_0\sin(\Omega t + \phi_0),$$
$$M_z = M\cos\theta_0,$$

where M, θ_0 and ϕ_0 are constants and $\Omega = \gamma B_0$. $\mathbf{M}$ is obviously an analytic function of γ and we can substitute $\gamma/(1 - \mathrm{i}\alpha)$ for γ in $\mathbf{M}$ to define the vector $\mathbf{N}$. This gives us

$$\xi = \tan\frac{\theta_0}{2}\,\mathrm{e}^{\mathrm{i}(\Omega' t+\phi_0)}\,\mathrm{e}^{-\alpha\Omega' t},$$

with $\Omega' = \Omega/(1+\alpha^2)$ and we recover the well-known solution of the LLG equation:

$$M'_x = M\frac{\sin\theta_0\cos(\Omega' t+\phi_0)}{\cosh(\alpha\Omega' t)+\cos\theta_0\sinh(\alpha\Omega' t)},$$
$$M'_y = M\frac{\sin\theta_0\sin(\Omega' t+\phi_0)}{\cosh(\alpha\Omega' t)+\cos\theta_0\sinh(\alpha\Omega' t)},$$
$$M'_z = M\frac{\cos\theta_0\cosh(\alpha\Omega' t)+\sinh(\alpha\Omega' t)}{\cosh(\alpha\Omega' t)+\cos\theta_0\sinh(\alpha\Omega' t)}.$$

If $\Omega > 0$, the equilibrium magnetization is $\mathbf{M}' = (0, 0, M)$; if $\Omega < 0$, it is $\mathbf{M}' = (0, 0, -M)$. As expected, damping transforms a precession dynamics into a motion towards an equilibrium state.

4. Two-level systems

It is convenient to determine directly the influence of damping on the dynamics of the two-level system described in the Schrödinger or Heisenberg picture. For a two-level system, the Schrödinger equation is

$$\mathrm{i}\hbar\frac{\mathrm{d}\psi}{\mathrm{d}t} = H(t)\psi,$$

where ψ has two components ψ_1 and ψ_2. The Hamiltonian can be written as $H(t) = (\hbar\gamma/2)\big(B_0(t) + \sum_j B_j(t)\sigma_j\big)$, where the constant γ has been added for later convenience and where σ_j are the Pauli matrices, so that $\sigma_a\sigma_b = \delta_{ab} + \mathrm{i}\epsilon_{abc}\sigma_c$. Defining $f(t) = \exp\big(-\mathrm{i}\gamma\int_0^t B_0(\tau)\,\mathrm{d}\tau\big)$ and $\psi(t) = f(t)\psi'(t)$ turn the Schrödinger equation for ψ into a Schrödinger equation for ψ' with the Hamiltonian $H = (\hbar\gamma/2)\sum_j B_j\sigma_j$. Thus, without loss of generality, we use the latter Hamiltonian.

Following Feynman *et al* [16], the relation between the density matrix ρ (with matrix elements $\rho_{ij} = \psi_i\psi_j^*$) and the magnetic moment $\mathbf{M}$ is $M_x = \rho_{12} + \rho_{21}$, $M_y = \mathrm{i}(\rho_{12} - \rho_{21})$ and $M_z = \rho_{11} - \rho_{22}$. Thus, we obtain $\rho = (1/2)\big(1 + \sum_j M_j\sigma_j\big)$ and $|\mathbf{M}| = 1$. We see that $\xi = (M_x + \mathrm{i}M_y)/(1 + M_z) = \psi_2/\psi_1$. If we diagonalize ρ, we recover the states ψ_1 and ψ_2 up to a phase that cannot be specified easily. Therefore, we shall work with the density matrix. The equation of motion for ρ is

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}[H, \rho].$$

The commutation relations for the Pauli matrices turn this equation into the Bloch equation

$$\dot{\mathbf{M}} = -\gamma\mathbf{M}\times\mathbf{B}.$$

To turn the Bloch equation into the LLG equation, we just replace $\mathbf{M}$ by $\mathbf{M}'$ and $\mathbf{B}$ by $\mathbf{B} - (\alpha/\gamma)\dot{\mathbf{M}}'$. If we denote by ρ' the density matrix corresponding to $\mathbf{M}'$, we find the equation of motion in the presence of damping

$$\dot\rho' = -\frac{\mathrm{i}}{\hbar}[H, \rho'] + \mathrm{i}\alpha[\dot\rho', \rho'].$$

In other words, the damping term of the LLG equation is transformed into a nonlinear term $\mathrm{i}\alpha[\dot\rho', \rho']$ in the equation of motion of the density matrix.

If we replace $\mathbf{B}$ by $\mathbf{B}-(\alpha/\gamma)\dot{\mathbf{M}}'$ in the Schrödinger equation itself, we obtain the nonlinear Schrödinger equation

$$\frac{\mathrm{d}\psi_1'}{\mathrm{d}t} = \big((1+2\mathrm{i}\alpha|\psi_2'|^2)B_3 - \mathrm{i}\alpha\psi_1(\psi_2')^*B_+\big)\psi_1' + \big(1+\mathrm{i}\alpha|\psi_2'|^2\big)B_-\psi_2',$$
$$\frac{\mathrm{d}\psi_2'}{\mathrm{d}t} = -\big((1+2\mathrm{i}\alpha|\psi_1'|^2)B_3 + \mathrm{i}\alpha(\psi_1')^*\psi_2'B_-\big)\psi_2' + \big(1+\mathrm{i}\alpha|\psi_1'|^2\big)B_+\psi_1'.$$

Therefore, our method transforms the analytic solution of a Bloch equation into the analytic solution of a nonlinear Schrödinger equation. We also reach the surprising conclusion that the nonlinear terms in the nonlinear Schrödinger equation can describe the influence of damping. However, this damping does not create decoherence: the transformed vector $\mathbf{M}'$ is still real and of length 1. As a consequence, $(\rho')^2 = \rho'$ and ρ' is the density matrix of a pure state.

5. Self-induced transparency

We investigate now the effect of damping on a famous nonperturbative effect in two-level systems: self-induced transparency.

McCall and Hahn [17] discovered a solution of the Bloch equation for the hyperbolic secant pulse $\mathbf{B} = (a/\cosh(t/\tau), 0, 0)$. We recall that $\int_{-\infty}^{\infty}\mathrm{d}t/\cosh(t/\tau) = \tau\pi$. We consider more generally a spin system submitted to a time-varying magnetic field linearly polarized along Ox: $\mathbf{B} = (b(t), 0, 0)$, where $b(t)$ is only required to be integrable. At time $t = t_0$, the spin has the spherical coordinates θ_0 and ϕ_0. Let $f(t) = \int_{t_0}^{t}\mathrm{d}\tau\, b(\tau)$ and $a = (1-\xi_0)/(1+\xi_0)$ with $\xi_0 = \tan(\theta_0/2)\,\mathrm{e}^{\mathrm{i}\phi_0}$. The solution of the Bloch equation is

$$M_x(t) = M\frac{1-\rho^2}{1+\rho^2},$$
$$M_y(t) = -2M\frac{\rho\sin(x(t)+\varphi)}{1+\rho^2},$$
$$M_z(t) = 2M\frac{\rho\cos(x(t)+\varphi)}{1+\rho^2},$$

where ρ and φ are the modulus and argument of a, respectively, and $x(t) = \gamma f(t)$. Self-induced transparency occurs when $x(\infty) = 2n\pi$ because, after a long interaction with the external field, the system finds itself back in its state at t_0. In the case of the McCall and Hahn pulse, we find $f(t) = 2a\tau\arctan(\tanh((t-t_0)/2\tau))$, which tends to $a\tau\pi/2$ for large t. Therefore, self-induced transparency occurs when $\gamma a\tau = 4n$ for some integer n.

To determine the effect of damping on this phenomenon, we calculate

$$\xi = \frac{1-a\,\mathrm{e}^{\mathrm{i}\bar{\gamma}f(t)}}{1+a\,\mathrm{e}^{\mathrm{i}\bar{\gamma}f(t)}},$$

and the corresponding solution of the LLG equation is

$$M_x'(t) = M\frac{\mathrm{e}^{2\alpha\bar{x}(t)}-\rho^2}{\mathrm{e}^{2\alpha\bar{x}(t)}+\rho^2},$$
$$M_y'(t) = -2M\frac{\mathrm{e}^{\alpha\bar{x}(t)}\rho\sin(\bar{x}(t)+\varphi)}{\mathrm{e}^{2\alpha\bar{x}(t)}+\rho^2},$$
$$M_z'(t) = 2M\frac{\mathrm{e}^{\alpha\bar{x}(t)}\rho\cos(\bar{x}(t)+\varphi)}{\mathrm{e}^{2\alpha\bar{x}(t)}+\rho^2},$$

with $\bar{x}(t) = \gamma f(t)/(1+\alpha^2)$. Two effects can be observed. First, the resonance condition is shifted from $x(\infty) = 2n\pi$ to $x(\infty) = 2n\pi(1+\alpha^2)$; second, even at resonance the initial state is not fully recovered. For example, if the system is initially in the state $\psi_1(0) = 1$, $\psi_2(0) = 0$, we have $M = 1$, $\rho = 1$ and $\varphi = 0$ and the final state is, at resonance,

$$M'_x(\infty) = \tanh(\alpha\bar{x}(\infty)),$$
$$M'_y(\infty) = 0,$$
$$M'_z(\infty) = \frac{1}{\cosh(\alpha\bar{x}(\infty))}.$$

The initial state cannot be recovered because $\alpha\bar{x}(\infty) \neq 0$. Therefore, although the damping of the LLG equation amounts only to a trend towards an equilibrium state and not to a decoherence, it leads to a loss of self-induced transparency. However, if α is small, this loss is reasonably small.

Exact solutions describing self-induced transparency in the presence of an amplitude- and frequency-modulated pulse were discovered by Hioe [18]. The transformation applies to this more general case.

6. Conclusion

It was shown in this paper that analytic solutions of the Bloch equations can be transformed into analytic solutions of the LLG equation. The influence of damping leads to interesting results for two-level systems, in particular the transformation of the linear into the nonlinear Schrödinger equation. As an application, we provided an exact description of self-induced transparency in the presence of damping.

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